Pergamon

Int. J. Solids Structures Vol. 31, No. 24, pp. 3387–3405, 1994

0020–7683/94 $7.00 + .00

0020–7683(94)00091–3

MIXTURE MODELING OF JOINTED FLUID-SATURATED POROUS MEDIA

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(*Received* 17 *May* 1993; *in revised form* 18 *May* 1994)

Abstract—The quasi-static equations of motion are studied for bi-laminated fluid-saturated porous media within the framework of non-phenomenological mixture theories. The flow–deformation coupled behavior of the media is governed by Biot's theory for which all constituents are considered compressible. The asymptotic analysis for a periodic microstructure with multiple scales, developed by Hegemier and Murakami, is adopted to obtain the equations of equilibrium and mass conservation in a binary saturated porous medium. The multiscale analysis appears to be advantageous for dealing with consolidation phenomena because it is capable of transforming a coupled, transient problem into two decoupled, steady-state ones. Various models with different degrees of approximation are generated, and among them a theory for saturated rocks with a single joint system is described. Mixture properties are expressed explicitly in terms of characteristics of intact and joint material. The most distinctive feature of this model comes from the fact that some cross terms, that have not been included in previous models, appear in the constitutive equations for fluid mass change and fluid flux. These cross terms are physically understood because they simply take into account effects occurring on the local level: the deformation–flow coupled phenomenon, the stress continuity and displacement compatibility conditions. These novel results may have far-reaching consequences for future theoretical modeling and experimental programs in two-phase fluid-filled porous media.

1. INTRODUCTION

Study of fluid-saturated porous media is of great interest for geotechnicians, geophysicists, geohydrologists and material scientists because a broad range of materials, such as soils, rocks, biomaterials and sol-gel materials, fall into the category of porous media. One of the main foci of study on these materials is the so-called consolidation problem, which is concerned with the coupled behavior between fluid flow and solid deformation pattern, the constituents being compressible or incompressible. The imposed excitations may be static or dynamic. The one-dimensional theory of consolidation, proposed by Terzaghi [see, for example, Terzaghi (1948)], and its three-dimensional counterpart, developed by Biot (1941), provide a rational basis to interpret field and laboratory observations, predict performance of structures and devices, and extrapolate science engineering experiences from one case to another. As the classical theories were formulated to model homogeneous media, their applicability to nonhomogeneous media has to be accompanied by a numerical procedure like the finite element method.

Among many types of heterogeneity that may exist in a continuous medium, discontinuities such as fissures, cracks, veins and joints are found frequently. A typical medium with discontinuities is a rock mass with joints. Large-scale discontinuities like bedding surfaces and joints are characteristic features of rock masses. Because joints are generally weaker and more permeable than an intact rock mass, careful consideration of them is of vital importance in assessing slope stability, excavation design, mine drainage system design and development of groundwater resources.

As early as the 1960s, joint elements were introduced (Goodman *et al.*, 1968; Pande *et al.*, 1990) to model the sharp change of displacement across joint surfaces in the finite element scheme. However, this direct modeling of discontinuities in a continuum is not always feasible. For a finite element analysis, in order that the variations of displacement and stress across the joints are accurately captured, it is necessary to employ several elements between joints. If the problem domain contains a great number of joints, the total problem size quickly escalates; such a rigorous analysis becomes impossible even with the aid of

supercomputers, or may be possible but economically unaffordable in practice. The same computational difficulties also arise if the boundary integral element method is used.

There is one alternative way to overcome the above mentioned numerical difficulties and represent the complicated behavior of the jointed media correctly. It consists of mathematically homogenizing the heterogeneous media to obtain motion and constitutive equations for equivalent homogeneous media first and then using the homogenized equations to solve a boundary value problem with the boundary conditions originally imposed. Adopting this continuum approach and dealing with flow problems in a medium that is randomly jointed and undeformable, Barenblatt [see Barenblatt *et al.*, 1990)] developed, in 1960, a so-called double porosity model. This model, which is now widely used in petroleum applications, allows consideration of the distinctive two-flow path, one through pores and another by joints. As to deformation behavior, Singh (1973) derived constitutive equations in an attempt to characterize regularly jointed rock masses from a continuum viewpoint in which no fluid flow was considered. A theory for flow–deformation coupled behavior had not been synthesized until very recently.

One of the early attempts to model the deformation–flow behaviour of jointed/fractured fluid-saturated porous media was made by Aifantis and his co-workers (Wilson and Aifantis, 1982; Khaled *et al.*, 1984; Beskos and Aifantis, 1986). Aifantis' model is of a phenomenological type. The motion equations for a random medium are derived from a macroscopic viewpoint; the coefficients involved in the constitutive equations have to be determined by experiments. Although this model has been demonstrated to be physically sound, mathematically correct and applicable in many practical situations (Cho *et al.*, 1991; Elsworth and Bai, 1992), its use seems weakened by two limitations. First, the fact that all the model coefficients must be experimentally determined limits, to some extent, its predictive capability. In fact, the size of joint systems in rock mass may be so large that a direct experimental evaluation of the overall behavior of jointed media is impossible. Under these circumstances, it is more practical to analytically evaluate the overall response of the media based on individual knowledge about the intact and joint materials. In addition, as basic equations are derived by making macroscopic assumptions, it is difficult to assess their validity. Hence, it is desirable to study this problem from another theoretical viewpoint.

This paper aims at formulating a theory for a through-jointed saturated porous medium from a non-phenomenological viewpoint, different from those presented in previous research works. The material is considered essentially elastic and no inertia effects are included. The overall behavior of the jointed media is determined explicitly in terms of the local deformation and flow characteristics of joint and intact materials. Because no *a priori* assumptions will be made on the macroscopic behavior of the media, it is expected that all relevant effects are included and the model displays a complete picture of the mixture behavior. In fact, it will be demonstrated that some important effects have previously been discarded.

The model will be synthesized in a mixture representation in such a way that three average quantities are assigned to a spatial position: solid skeleton displacement, pore pressure and joint pressure. The general methodology developed by Hegemier, Murakami and their collaborators (Murakami *et al.*, 1981; Murakami, 1985; Murakami and Hegemier, 1989; Murakami and Toledano, 1990) was followed for obtaining mixture equations. The Hegemier–Murakami approach has been elaborated on in the past to study wave propagation phenomena in composite materials. The method consists of the following three main stages. First, the media under consideration are regularly periodic so one can choose two distinctive length scales, the macroscopic wavelength and the dimension of periodic cell. For these periodic media a multiscale representation is carried out. Second, a volume average procedure is employed to obtain the motion and constitutive equations in the average sense, from which new independent variables emerge. Third, due to the presence of new variables, new equations are required which are synthesized from the modified Reissner's variational principle. The trial fields necessary for this variational principle are inferred from an asymptotic analysis. This non-phenomenological mixture theory has been used to study composites with different geometries (sphere, cylinder and laminate) and the results are encouraging in comparison with experimental observations.

The Hegemier–Murakami method will be used because it takes into account the dimension of periodic cell. It has been shown that such micro-dependence plays an important role in modeling wave attenuation and dispersion phenomena in composites. The need for including the micro-dependence even in a quasi-static consolidation problem will be demonstrated in this paper. Although the above described method will be followed, there exist some differences between previous research works and the present paper. The media under study herein are fluid-saturated porous media instead of the nonporous solid materials analysed before, so a theory for porous media such as Biot's formulation should be used. In addition, some simplifications have to be made on the general framework established in the previous works, because we want to obtain a theory consistent with phenomenological models. Specifically, the high-order variables previously used will be eliminated in a systematic way.

For the sake of simplicity in this work, a porous medium with a single persistent joint system has been chosen. However, many conclusions drawn here are also valid for a medium with more complicated discontinuities, for example, a medium with a non-persistent joint system or simply a random medium. The present model is also easily extended to treat nonlinear problems if it is adopted in an incremental form. The key contribution of this paper on the subject is two fold. A mixture theory is presented for a special type of nonhomogeneous fluid-saturated porous media that is of great interest in engineering practice and, more importantly, some novel deformation–flow coupling phenomena in a binary saturated porous medium are revealed.

The work is organized as follows. In the next section, Biot's basic equations are presented and multiscale representation of the original problem is introduced. Although this work is concerned with a particular model for jointed media, for the sake of clarity, a general mixture model for bi-laminated media is first presented in Section 3. This model is subsequently specialized to jointed media in Section 4, where final motion and constitutive equations are obtained. We give a comprehensive discussion of the model in Section 5. Indicial notation is used throughout the paper and the summation convention is employed with δ_{ij} being Kronecker's delta. Before closing the Introduction, it should be noted that, in this paper, microscopic and macroscopic variables refer to the observed values of those variables when the media are seen as homogeneous and heterogeneous, respectively. Therefore, the distinction between the two types of variables is merely due to the different length scales.

2. MULTISCALE REPRESENTATION OF BINARY MEDIA

The geometry of the problem is shown in Fig. 1. We deal with a periodic medium. The periodic cells are formed in the direction x_1 and have the dimension l. If L denotes the macroscopic wavelength, the nondimensional periodicity for the cell is defined as

$$\varepsilon = \frac{l}{L}. \tag{1}$$

The isolated central part of the cell is occupied by the fluid-saturated porous material 1. The region of the material 1 (or 2) is not interconnected with the other region of the same material so one has to pass across the contact area from one material to another.

The motion equations for each fluid-saturated porous material are those presented by Biot (1941). The motion of any material point depends exclusively upon its spatial position and time, respectively denoted by $\mathbf{x}$ and t. For the sake of clarity, the inertia effect is neglected but its consideration can be made in a straightforward manner. At this stage, two special features may be noted in the media. Its geometrical array possesses special periodicity measured by ε; the two materials within each cell are characterized by different mechanical and hydraulic properties. So each field quantity is to be specialized by using the superindices ε and (α) where $\alpha = 1, 2$. The static equilibrium equation and mass conservation equation are then given by

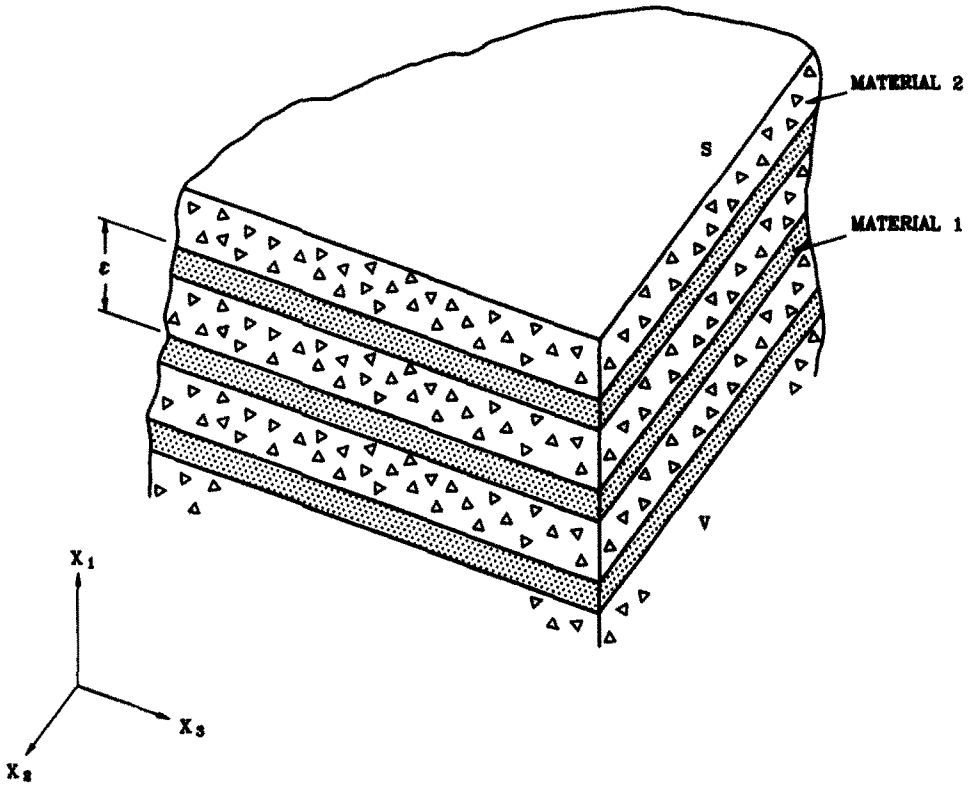

Fig. 1. Schematic view of bi-laminated porous media at the macroscopic level.

$$\sigma_{ji}^{(\alpha)\varepsilon}(\mathbf{x}, t)_{,j} = 0$$
$$\dot{m}^{(\alpha)\varepsilon}(\mathbf{x}, t) + q_i^{(\alpha)\varepsilon}(\mathbf{x}, t)_{,i} = 0, \tag{2}$$

where σ_{ji} is the total stress tensor, $\dot{m}$ is the rate of fluid mass change defined in an undeformed state, q_i is the fluid flux vector and $\mathbf{x}$ is the macrocoordinate. $\dot{(\ )}$ and $(\ )_{,j}$ denote the time and spatial derivatives, respectively.

The motion eqns (2) are complemented by a set of constitutive equations for each material. The first of them is the relation between the total stress–strain of the solid skeleton and the pore pressure, as

$$\sigma_{ij}^{(\alpha)\varepsilon}(\mathbf{x}, t) = C_{ijkl}^{(\alpha)}(\mathbf{x})\, e_{kl}^{(\alpha)\varepsilon}(\mathbf{x}, t) - \zeta^{(\alpha)}(\mathbf{x}) p^{(\alpha)\varepsilon}(\mathbf{x}, t)\delta_{ij}, \tag{3}$$

where the deformation tensor is defined as

$$2e_{ij}^{(\alpha)\varepsilon}(\mathbf{x}, t) = u_i^{(\alpha)\varepsilon}(\mathbf{x}, t)_{,j} + u_j^{(\alpha)\varepsilon}(\mathbf{x}, t)_{,i}.$$

C_{ijkl} is the stiffness tensor of the solid skeleton that may represent the nonlinearity and anisotropy of the solid skeleton behavior. The scalar quantity $\zeta^{(\alpha)}$ can be expressed as

$$\zeta^{(\alpha)} = 1 - \left(\frac{K}{K_s'}\right)^{(\alpha)} \tag{4}$$

in terms of the stiffness parameters for solid skeleton and unjacketed volume, K and K_s' [see, for example, Rice and Cleary (1976)]. The second constitutive equation dictates the fluid mass change in terms of the volumetric strain and pore pressure, given by

$$m^{(\alpha)\varepsilon}(\mathbf{x}, t) = \zeta^{(\alpha)}(\mathbf{x}) e_{kk}^{(\alpha)\varepsilon}(\mathbf{x}, t) + \eta^{(\alpha)}(\mathbf{x}) p^{(\alpha)\varepsilon}(\mathbf{x}, t), \tag{5}$$

where the constant $\eta^{(\alpha)}$ is expressed by

$$\eta^{(\alpha)} = \left(\frac{\phi_f}{K_f} + \frac{\zeta}{K_s'} - \frac{\phi_f}{K_s''} \right)^{(\alpha)} \tag{6}$$

in terms of the stiffness parameters for unjacketed pore and pore fluid K_s'' and K_f, respectively. In eqn (6), ϕ_f is the porosity of each constituent, which must not be confused with the volume fraction $\phi^{(\alpha)}$ that will be defined in the following section. The third and last constitutive relation is established between the fluid flux and the pore pressure gradient, known as Darcy's law:

$$q_i^{(\alpha)\varepsilon}(\mathbf{x}, t) = -\kappa_{ij}^{(\alpha)}(\mathbf{x}) p^{(\alpha)\varepsilon}(\mathbf{x}, t)_{,j}. \tag{7}$$

The permeability tensor $\kappa_{ij}^{(\alpha)}$ defined in eqn (7) may be related to the coefficients of permeability k_{ij} and the fluid viscosity μ as $\kappa_{ij}^{(\alpha)} = (k_{ij}/\mu)^{(\alpha)}$.

In the constitutive equations (3) and (5), both solid and fluid phases are considered compressible, containing distinctive stiffness values that should be determined experimentally. However, this general form of constitutive equations may be simplified for practical purposes. It is generally assumed that the unjacketed volume and pore stiffness take the same value as solid grain stiffness K_s, i.e. $K_s' = K_s'' = K_s$. Also, if the compressibility of solid grain is small compared with that of solid skeleton, we have $\zeta = 1$ and $\eta = \phi_f/K_f$. If the compressibility of the pore fluid is neglected, we further have $\eta = 0$, implying that the pore pressure does not make any contribution to the fluid mass change. We can see that constants ζ and η are always positive; the first one's upper bound is 1 and the second one's lower bound is 0. Note also that as the stiffness parameter K_f is used to characterize the pore fluid, the fluid is then considered herein as an elastic, inviscid material.

Finally, the boundary conditions are given by

$$\begin{aligned}
&u_i^{(\alpha)\varepsilon}(\mathbf{x}, t) = U_i^{(\alpha)\varepsilon}(\mathbf{x}, t) && \text{on } \Gamma_u^{(\alpha)} \\
&\sigma_{ij}^{(\alpha)\varepsilon}(\mathbf{x}, t)\, n_j(\mathbf{x}) = S_i^{(\alpha)\varepsilon}(\mathbf{x}, t) && \text{on } \Gamma_s^{(\alpha)} \\
&p^{(\alpha)\varepsilon}(\mathbf{x}, t) = P^{(\alpha)\varepsilon}(\mathbf{x}, t) && \text{on } \Gamma_p^{(\alpha)} \\
&q_i^{(\alpha)\varepsilon}(\mathbf{x}, t)\, n_i(\mathbf{x}) = Q^{(\alpha)\varepsilon}(\mathbf{x}, t) && \text{on } \Gamma_q^{(\alpha)}
\end{aligned} \tag{8}$$

with prescribed values of displacement $U_i^{(\alpha)\varepsilon}$, traction $S_i^{(\alpha)\varepsilon}$, fluid pressure $P^{(\alpha)\varepsilon}$ and fluid flux $Q^{(\alpha)\varepsilon}$. The vector n_i is the outward normal of the domain Ω on the surface $\Gamma_s^{(\alpha)}$ or $\Gamma_q^{(\alpha)}$.

Equations (2), (3), (5), (7) and (8), along with initial conditions for $u_i^{(\alpha)\varepsilon}$ and $p^{(\alpha)\varepsilon}$, define a well posed initial boundary value problem that, however, is difficult to solve if the number of cells is great. The most important computational task arises from the fact that the domain under consideration has two very different length scales. In order to make the problem numerically manageable, we have to distinguish two length scales first and seek some methods to eliminate the dependence of the smaller length scale. This may be accomplished by a multivariable representation.

Let us consider the cell that, as described above, consists of a complete region of material 1 and two adjacent half regions of material 2 (Fig. 2). Within the cell, all the field quantities continue to depend upon the spatial position in the macrocoordinate system $\mathbf{x}$, and they may also be seen as a dependent function of the spatial position in a microcoordinate system that may be established for the cell. The microcoordinate in the periodicity direction x_1 will be obtained through scaling x_1 by the periodicity ε

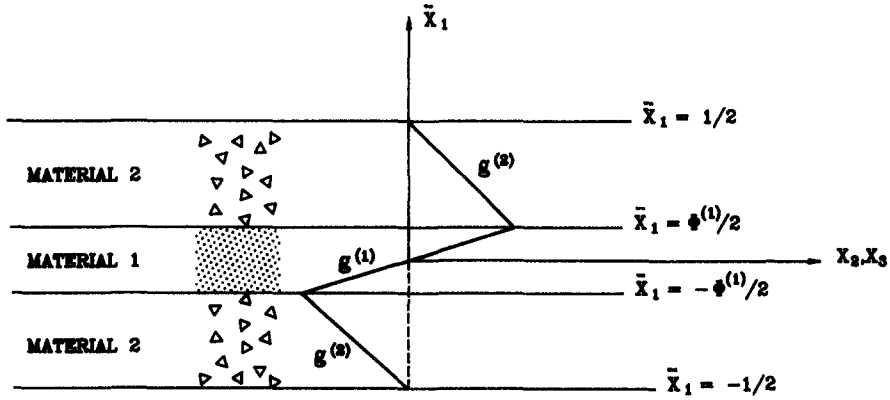

Fig. 2. Schematic view of bi-laminated porous media at the microscopic level.

$$\bar{x}_1 = \frac{x_1}{\varepsilon L} = \frac{x_1}{l}. \tag{9}$$

The nondimensional coordinate $\bar{x}_1$ varies from $-1/2$ to $1/2$ in a cell. Whereas the spatial scaling is made, no temporal counterpart will be introduced. This implies that the elastic characteristics of the filling fluid will be preserved. Once the length scaling change is made, the original field variables expressed in eqns (2), (3), (5) and (7) can be transformed to those in both macroscopic and microscopic coordinate systems:

$$\sigma_{ji}^{(\alpha)\varepsilon}(\mathbf{x}, \bar{x}_1, t)_{,j} + \frac{1}{l}\sigma_{1i}^{(\alpha)\varepsilon}(\mathbf{x}, \bar{x}_1, t)_{,\bar{1}} = 0$$

$$\dot{m}^{(\alpha)\varepsilon}(\mathbf{x}, \bar{x}_1, t) + q_i^{(\alpha)\varepsilon}(\mathbf{x}, \bar{x}_1, t)_{,i} + \frac{1}{l}q_1^{(\alpha)\varepsilon}(\mathbf{x}, \bar{x}_1, t)_{,\bar{1}} = 0$$

$$\sigma_{ij}^{(\alpha)\varepsilon}(\mathbf{x}, \bar{x}_1, t) = C_{ijkl}^{(\alpha)}(\mathbf{x})\left(e_{kl}^{(\alpha)\varepsilon}(\mathbf{x}, \bar{x}_1, t) + \frac{1}{l}\bar{e}_{kl}^{(\alpha)\varepsilon}(\mathbf{x}, \bar{x}_1, t)\right) - \zeta^{(\alpha)}(\mathbf{x})p^{(\alpha)\varepsilon}(\mathbf{x}, \bar{x}_1, t)\delta_{ij}$$

$$m^{(\alpha)\varepsilon}(\mathbf{x}, \bar{x}_1, t) = \zeta^{(\alpha)}(\mathbf{x})\left(e_{kk}^{(\alpha)\varepsilon}(\mathbf{x}, \bar{x}_1, t) + \frac{1}{l}\bar{e}_{kk}^{(\alpha)\varepsilon}(\mathbf{x}, \bar{x}_1, t)\right) + \eta^{(\alpha)}(\mathbf{x})p^{(\alpha)\varepsilon}(\mathbf{x}, \bar{x}_1, t)$$

$$q_i^{(\alpha)\varepsilon}(\mathbf{x}, \bar{x}_1, t) = -\kappa_{ij}^{(\alpha)}(\mathbf{x})\left(p^{(\alpha)\varepsilon}(\mathbf{x}, \bar{x}_1, t)_{,j} + \frac{1}{l}\delta_{j1}p^{(\alpha)\varepsilon}(\mathbf{x}, \bar{x}_1, t)_{,\bar{1}}\right), \tag{10}$$

where

$$2\bar{e}_{ij}^{(\alpha)\varepsilon}(\mathbf{x}, \bar{x}_1, t) = \delta_{j1}u_i^{(\alpha)\varepsilon}(\mathbf{x}, \bar{x}_1, t)_{,\bar{1}} + \delta_{i1}u_j^{(\alpha)\varepsilon}(\mathbf{x}, \bar{x}_1, t)_{,\bar{1}}$$

and the subindex $\bar{1}$ denotes the microcoordinate $\bar{x}_1$. Note that under the multivariable transformation, all the field quantities depend not only on the macrocoordinate but also on the microcoordinate. This, however, is not true for the material properties such as $C_{ijkl}^{(\alpha)}$, $\kappa_{ij}^{(\alpha)}$, $\zeta^{(\alpha)}$ and $\eta^{(\alpha)}$, implying that they are assumed to be homogeneous in each region of the cell. Note also that the initial and boundary conditions also become dependent on both coordinate systems.

With the multiscale representation, the spatial derivative of field variables is split into two parts, macroscopic and microscopic. This is very helpful for the volume average procedure as we will see in the following section.

3. MIXTURE ANALYSIS OF BI-LAMINATED MEDIA

The micro-dependence of field quantities may be eliminated by following a volume average procedure. An average quantity ψ is defined as

$$\psi^{(\alpha)}(\mathbf{x}, t) \equiv <\psi^{(\alpha)\varepsilon}> = \frac{1}{\phi^{(\alpha)}} \int_{v^{(\alpha)}} \psi^{(\alpha)\varepsilon}(\mathbf{x}, \bar{x}_1, t)\, d\bar{x}_1, \tag{11}$$

in which ψ represents one of the following quantities: u_i, σ_{ij}, p, q or m. $\phi^{(\alpha)}$ is the volume fraction and $v^{(\alpha)}$ is the microdomain for each phase. The symbol $< >$ means intrinsic volume averaging.

The volume average, as defined by eqn (11), can be used not only to calculate average values of field variables, but also to obtain average motion equations. This procedure starts from eqns $(10)_1$ and $(10)_2$. As a result of the multiscale representation described in the previous section, the micro- and macro-dependencies of spatial derivatives have been separated in these equations. For example, in eqn $(10)_1$, the first term is exclusively related to the derivative with respect to the macrocoordinate, and the second term, to the derivative with respect to the microcoordinate. Consequently, for the first term in eqn (10), the average of the derivative is simply the derivative of the average. Therefore, the average procedure can be carried out in a relatively simple way. This is the main advantage of the multiscale representation employed in the last section. Using this line of reasoning and applying eqn (11), the average forms of eqns $(10)_1$ and $(10)_2$ become

$$\begin{aligned} \phi^{(\alpha)}\sigma^{(\alpha)}_{ji,j} + (-1)^{\alpha+1}\bar{\sigma}_i &= 0 \\ \phi^{(\alpha)}\dot{m}^{(\alpha)} + \phi^{(\alpha)}q^{(\alpha)}_{i,i} + (-1)^{\alpha+1}\bar{q} &= 0, \end{aligned} \tag{12}$$

where two interaction quantities $\bar{\phi}_i$ and $\bar{q}$ are introduced:

$$\begin{aligned} \bar{\sigma}_i(\mathbf{x}, t) &= \frac{1}{l}\left(\sigma^{(\alpha)\varepsilon}_{1i}(\bar{x}_1 = \phi^{(1)}/2) - \sigma^{(\alpha)\varepsilon}_{1i}(\bar{x}_1 = -\phi^{(1)}/2)\right) \\ \bar{q}(\mathbf{x}, t) &= \frac{1}{l}\left(q^{(\alpha)\varepsilon}_{1}(\bar{x}_1 = \phi^{(1)}/2) - q^{(\alpha)\varepsilon}_{1}(\bar{x}_1 = -\phi^{(1)}/2)\right). \end{aligned} \tag{13}$$

These two quantities measure transverse stresses and transverse flux along the interfacial plane between the two constituents. They are macro-dependent quantities but represent the microstructural features of the problem. In defining eqn (13), it is assumed that the transverse stresses and flux are continuous across the interfacial area, so the terms labeled (α) on the right-hand side of eqn (13) may be evaluated for any value of α. In this work, $\bar{\sigma}_i$ and $\bar{q}$ are called microstress and microflux, respectively, due to their resemblance with those microquantities introduced by Mindlin [see, for example, Bedford and Drumheller (1983)] in his continuum theory with microstructure.

Likewise, the average constitutive equations are obtained from eqns $(10)_3$–$(10)_5$ as

$$\begin{aligned} \sigma^{(\alpha)}_{ij} &= C^{(\alpha)}_{ijkl}e^{(\alpha)}_{kl} + (-1)^{\alpha+1}C^{(\alpha)}_{ijk1}\bar{u}_k/\phi^{(\alpha)} - \zeta^{(\alpha)}p^{(\alpha)}\delta_{ij} \\ m^{(\alpha)} &= \zeta^{(\alpha)}e^{(\alpha)}_{kk} + (-1)^{\alpha+1}\zeta^{(\alpha)}\bar{u}_1/\phi^{(\alpha)} + \eta^{(\alpha)}p^{(\alpha)} \\ q^{(\alpha)}_i &= -\kappa^{(\alpha)}_{ij}p^{(\alpha)}_{,j} - (-1)^{\alpha+1}\kappa^{(\alpha)}_{i1}\bar{p}/\phi^{(\alpha)}, \end{aligned} \tag{14}$$

where the average deformation tensor is defined by

$$2e^{(\alpha)}_{ij} = u^{(\alpha)}_{i,j} + u^{(\alpha)}_{j,i}.$$

In eqn (14) two new kinematic quantities appear, defined by

$$\bar{u}_i(\mathbf{x}, t) = \frac{1}{l}(u_i^{(\alpha)\varepsilon}(\bar{x}_1 = \phi^{(1)}/2) - u_i^{(\alpha)\varepsilon}(\bar{x}_1 = -\phi^{(1)}/2))$$

$$\bar{p}(\mathbf{x}, t) = \frac{1}{l}(p^{(\alpha)\varepsilon}(\bar{x}_1 = \phi^{(1)}/2) - p^{(\alpha)\varepsilon}(\bar{x}_1 = -\phi^{(1)}/2)). \tag{15}$$

$\bar{u}_i$ and $\bar{p}$ measure the differences between interfacial displacements and interfacial pore pressures, respectively, and are macro-dependent quantities, just like $\bar{\sigma}_i$ and $\bar{q}$, but they are all closely related to the microstructure. $\bar{u}_i$ and $\bar{p}$ are called microdisplacement and micropressure, respectively. Finally, in the average context, the boundary conditions are defined by

$$\begin{aligned} u_i^{(\alpha)} &= U_i^{(\alpha)} \quad \text{on } \Gamma_u^{(\alpha)} \\ \sigma_{ij}^{(\alpha)} n_j &= S_i^{(\alpha)} \quad \text{on } \Gamma_s^{(\alpha)} \\ p^{(\alpha)} &= P^{(\alpha)} \quad \text{on } \Gamma_p^{(\alpha)} \\ q_i^{(\alpha)} n_i &= Q^{(\alpha)} \quad \text{on } \Gamma_q^{(\alpha)}. \end{aligned} \tag{16}$$

The motion equations (12), constitutive equations (14) and boundary conditions (16), along with initial conditions for kinematic quantities $u_i^{(\alpha)}, p^{(\alpha)}, \bar{u}_i$ and $\bar{p}$, define a well posed initial boundary problem for a homogeneous mixture. It is recalled that the original problem is nonhomogeneous because of the presence of two distinct periodic layers. However, the nonhomogeneity of the problem has been removed by an averaging procedure and a nonhomogeneous medium is represented now by an equivalent homogeneous one. Nevertheless, our task in the present work is far from being completed because new quantities like $\bar{\sigma}_i$, $\bar{q}$, $\bar{u}_i$ and $\bar{p}$, found in eqns (12) and (14), are as yet unknown. In order to find expressions for these quantities, we need to know the response of each cell at the microstructural level.

One efficient way to construct a solution for the cell is asymptotic analysis. If the nondimensional measure of the cell ε is small compared to one, the field variables may be expressed as asymptotic series as follows:

$$\psi^{(\alpha)\varepsilon}(\mathbf{x}, \bar{x}_1, t) = \psi_{(0)}^{(\alpha)}(\mathbf{x}, \bar{x}_1, t) + \varepsilon\, \psi_{(1)}^{(\alpha)}(\mathbf{x}, \bar{x}_1, t) + 0(\varepsilon^2), \tag{17}$$

where ψ represents u_i, σ_{ij}, q_i, p or m. Applying eqn (17) in eqn (10) and equating terms of equal ε powers, we obtain a series of equations that define the micro initial boundary value problems. For the lowest order $1/\varepsilon^2$, it is given that

$$u_{i(0),\bar{1}}^{(\alpha)} = 0, \qquad p_{(0),\bar{1}}^{(\alpha)} = 0 \tag{18}$$

and for $1/\varepsilon$,

$$\begin{aligned} \sigma_{1i(0),\bar{1}}^{(\alpha)} &= 0, \qquad q_{1(0),\bar{1}}^{(\alpha)} = 0 \\ \sigma_{ji(0)}^{(\alpha)} &= C_{jikl}^{(\alpha)} \tfrac{1}{2}(u_{1(0),k}^{(\alpha)} + u_{k(0),1}^{(\alpha)} + \delta_{k1} u_{l(1),\bar{1}}^{(\alpha)} + \delta_{l1} u_{k(1),\bar{1}}^{(\alpha)}) - \zeta^{(\alpha)} p_{(0)}^{(\alpha)} \delta_{ij} \\ q_{i(0)}^{(\alpha)} &= -\kappa_{ij}^{(\alpha)}(p_{(0),j}^{(\alpha)} + \delta_{j1} p_{(1),\bar{1}}^{(\alpha)}) \\ m_{(0)}^{(\alpha)} &= \zeta^{(\alpha)}(u_{k(0),k}^{(\alpha)} + u_{1(1),\bar{1}}^{(\alpha)}) - \eta^{(\alpha)} p_{(0)}^{(\alpha)}. \end{aligned} \tag{19}$$

Now the solutions for the micro problems can be found by solving a series of micro subproblems and superimposing the subsolutions.

The solutions for the micro initial boundary problems may be carried out by considering, for all the field quantities, both periodicity and compatibility conditions. Solutions for the first micro boundary value problem described by the set of eqns (18) are obvious:

$$\psi_{(0)}^{(\alpha)}(\mathbf{x}, \bar{x}_1, t) = \psi_{(0)}^{(\alpha)}(\mathbf{x}, t), \tag{20}$$

where ψ represents u_i and p. Equation (20) shows that the displacement and the pore pressure of the lowest order, $u_{i(0)}^{(\alpha)}$ and $p_{(0)}^{(\alpha)}$, depend only upon the macrocoordinate system $\mathbf{x}$ and the time t.

For solving the problem of order $1/\varepsilon$, it is interesting to observe that the original flow–deformation coupled problem has been decoupled, that is, $u_{i(1)}^{(\alpha)}$ and $p_{(1)}^{(\alpha)}$ can be solved separately, either by eqns $(19)_1$ and $(19)_3$ or $(19)_2$ and $(19)_4$, respectively. This is due to the fact that the spatial variation of the deformation of solid skeleton is always higher than that of the pore pressure by one order. Another advantageous result is that, if terms of first order are retained, the steady-state flow problem is the only one that must be solved. The solution for a more complicated transient flow problem can be avoided because the term related to the rate of change of fluid mass $m^{(\alpha)}$ does not appear in eqn $(19)_2$. These results are quite fortunate because many closed-form solutions for elasticity and steady-state conduction problems are available and can be combined to furnish final solutions to micro problems.

With respect to the solution for the displacement field, as suggested by eqns $(19)_1$ and $(19)_3$, $u_{i(1)}^{(\alpha)}$ is proportional to the gradient of displacement of the lowest order $u_{i(0),j}^{(\alpha)}$. Although the exact field for $u_{i(1)}^{(\alpha)}$ can be obtained, the expression is quite lengthy. Murakami and Hegemier (1989) suggest, alternatively, the use of a trial field instead of the exact one. In the context of this paper, the trial field for the displacement, denoted by $\hat{u}_i^{(\alpha)}$, should fulfill the following two requirements: its average should equal the mixture average $u_i^{(\alpha)}$ and, in view of the definition $(15)_1$, $\hat{u}_i^{(\alpha)}$ should be related to $\bar{u}_i$. In taking these requirements into account, we can propose the trial field for the displacement as follows:

$$u_i^{(\alpha)\varepsilon}(\mathbf{x}, \bar{x}_1, t) \approx \hat{u}_i^{(\alpha)}(\mathbf{x}, \bar{x}_1, t) = u_i^{(\alpha)}(\mathbf{x}, t) + l\bar{u}_i(\mathbf{x}, t)\, g^{(\alpha)}(\bar{x}_1), \tag{21}$$

where $g^{(\alpha)}(\bar{x}_1)$ is a linear function of $\bar{x}_1$

$$\begin{aligned} g^{(1)}(\bar{x}_1) &= \bar{x}_1/\phi^{(1)} \\ g^{(2)}(\bar{x}_1) &= (\operatorname{sgn}(\bar{x}_1)/2 - \bar{x}_1)/\phi^{(2)} \end{aligned} \tag{22}$$

for which $\operatorname{sgn}(\bar{x}_1) = -1, 0, 1$ for $\bar{x}_1 < 0, = 0$ and > 0. The form of the function $g^{(\alpha)}$ is shown in Fig. 2. Similar trial fields for fluid pressure, stress and flux can be proposed as follows:

$$\begin{aligned} \hat{p}^{(\alpha)}(\mathbf{x}, \bar{x}_1, t) &= p^{(\alpha)}(\mathbf{x}, t) + l\bar{p}(\mathbf{x}, t)\, g^{(\alpha)}(\bar{x}_1) \\ \hat{\sigma}_{ij}^{(\alpha)}(\mathbf{x}, \bar{x}_1, t) &= \sigma_{ij}^{(\alpha)}(\mathbf{x}, t) + l\delta_{j\bar{1}}\bar{\sigma}_i(\mathbf{x}, t)\, g^{(\alpha)}(\bar{x}_1) \\ \hat{q}_i^{(\alpha)}(\mathbf{x}, \bar{x}_1, t) &= q_i^{(\alpha)}(\mathbf{x}, t) + l\delta_{i\bar{1}}\bar{q}(\mathbf{x}, t)\, g^{(\alpha)}(\bar{x}_1), \end{aligned} \tag{23}$$

where $\delta_{j\bar{1}} = 1$ when $j = 1$ and $\delta_{j\bar{1}} = 0$ otherwise. Note that the same function $g^{(\alpha)}$ can be used for all the trial fields.

Once the trial fields are constructed, a variational principle can be used to find new equations. Murakami and Hegemier (1989) suggested the modified Reissner's variational principle, because it can take into account the following special features of the medium: the multiscale nature, discontinuity of the average field variables across the interfacial area, and the variation of stress and flux. The principle, in the form of virtual work and for the consolidation problem under consideration, is given in eqn (A1) of Appendix A. Equation (A1) is formulated in terms of trial fields. There is a difference between the principle presented in eqn (A1) and that used by Murakami and Hegemier (1989). Here, the variation of stress in all directions is included in contrast to the previous work where only the variation in the transverse direction was considered. This slight modification has no physical implication; it only allows for an easier algebraic task because taking variations in all

directions can lead to obtaining the constitutive equations for all directions from the same variational principle.

Applying the trial fields given in eqns (21) and (23) into eqn (A2), all the mixture equations as well as boundary conditions can be obtained. To begin with, there are two equilibrium equations now, one given in eqn $(12)_1$ and the other taking the form

$$\frac{l^2}{12}\bar{\sigma}_{ij,1}+\sigma_{1i}^{(2)}-\sigma_{1i}^{(1)}=0. \tag{24}$$

This equation establishes the relation between the transverse stresses and the microstress vector. The appearance of the cell dimension l in the first term of eqn (24) clearly indicates that this equation represents an equilibrium condition of higher order than that of eqn $(12)_1$, which is the one commonly established in continuum mechanics. On the other hand, there are also two equations of mass conservation, one of them presented in eqn $(12)_2$ and the other given by

$$\frac{l^2}{12}(q_{,1}+\zeta\dot{\bar{u}}_{k,k}+\eta\dot{\bar{p}})+q_1^{(2)}-q_1^{(1)}=0, \tag{25}$$

with $\zeta=\sum_\alpha \phi^{(\alpha)}\zeta^{(\alpha)}$ and $\eta=\sum_\alpha \phi^{(\alpha)}\eta^{(\alpha)}$. It is noted that, in obtaining eqn (25), the constitutive equation (5) was used to express the rate of fluid mass change. Equation (25) is an evolution equation for the microdisplacement and the micropressure, which predicts the history of higher-order terms during the consolidation process. On the other hand, two new constitutive equations appear, also as a result of the variational principle (A2), which take the forms

$$\frac{l^2}{12}\left(\frac{1}{2}(\bar{u}_{1,i}+\bar{u}_{i,1})+M_{ij}\bar{\sigma}_j+N_i\bar{p}\right)+u_i^{(2)}-u_i^{(1)}=0$$

$$\frac{l^2}{12}\left(\bar{p}_{,1}+\frac{1}{k}\bar{q}\right)+p^{(2)}-p^{(1)}=0, \tag{26}$$

with

$$M_{ij}=\sum_\alpha \phi^{(\alpha)}(C^{-1})^{(\alpha)}_{ikkj},\quad N_i=\sum_\alpha \phi^{(\alpha)}(C^{-1})^{(\alpha)}_{i1kk}\zeta^{(\alpha)},\quad \frac{1}{k}=\sum_\alpha \phi^{(\alpha)}(\kappa^{-1})^{(\alpha)}_{11}.$$

k is the average permeability in the direction normal to lamina.

Now we have a complete set of equations. The independent variables are the displacement for each phase $u_i^{(\alpha)}$, the microdisplacement $\bar{u}_i$, the pore pressure for each phase $p^{(\alpha)}$ and the micropressure $\bar{p}$. In a three-dimensional problem, there are twelve variables. A total of twelve equations come from eqns $(12)_1$, (24), $(12)_2$ and (25). Part of the boundary conditions for the problem are defined in eqn (16). The rest, which are related to the microquantities, are obtained from the variational principle (A2):

$$\begin{aligned}
&\delta\bar{u}_i=0 && \text{on } \Gamma_u\\
&\frac{l^2}{12}\bar{\sigma}_j n_j=\sum_\alpha \phi^{(\alpha)} l<g^{(\alpha)}S_1^{(\alpha)}> && \text{on } \Gamma_\sigma\\
&\delta\bar{p}=0 && \text{on } \Gamma_p\\
&\frac{l^2}{12}\bar{q}n_1=\sum_\alpha \phi^{(\alpha)} l<g^{(\alpha)}Q^{(\alpha)}> && \text{on } \Gamma_\sigma.
\end{aligned} \tag{27}$$

Along with initial conditions for all basic variables, the problem is well posed and its solution can be found. Due to the great number of basic variables involved in the model, it seems that for most problems of practical interest a numerical solution is needed. This may be accomplished by the finite element method or the boundary integral element method. But it is worth emphasizing that the use of the finite element method in a homogenized medium is quite different from its use in the original heterogeneous material. The mesh needed in the homogenized medium is considerably less fine because it is not necessary to subdivide each periodic cell into finite elements. This is one of the attractive features of the present model.

Because there are so many basic variables involved in the mixture equations, the model developed in this section is indeed elaborate. However, its practical significance will be clearer when it is specifically applied to a jointed medium in the following section. Nevertheless, it is interesting at this stage to indicate the general outcome of the model by discussing which simplified models can come out of the more complicated theory. Compared with an ordinary theory of continuous media, the present model is of a higher-order type due to the fact that it considers: (1) the space and time variations of microdisplacement and micropressure, and (2) the difference between phase displacements as well as between phase pressures. Thus the model can be simplified by relaxing both considerations.

If the time and space variations of microdisplacement and micropressure vanish, i.e. $\bar{u}_{i,j} = 0, \bar{p}_{,j} = 0, \dot{\bar{u}}_i = 0$ and $\dot{\bar{p}} = 0$, we can eliminate the microdisplacement and micropressure from the list of basic variables. Although the detail of eliminating basic variables will be given in the next section for jointed media, we outline the main stages here. In accepting the above simplification, the microstress and microflux that appear in eqn (26) are no longer independent but can be calculated from the displacement difference and pressure difference $u_i^{(2)} - u_i^{(1)}, p^{(2)} - p^{(1)}$. Substituting these relations into eqns (24) and (25) and using eqns $(14)_1$ and $(14)_3$, the expressions for the microdisplacement and micropressure are obtained in terms of phase displacements and phase pressures. Therefore, the mixture eqns (12), resulting from a simple volume average procedure, form the motion equations and no other high-order equilibrium equations are needed. Further simplification can be made by equating the phase displacements and phase pressures, that is, $u_i^{(1)} = u_i^{(2)} = u_i$ and $p^{(1)} = p^{(2)} = p$. In doing so, a simple consolidation theory with only four basic variables is obtained. The coefficients that emerge from this simple model are the so-called effective properties.

Between the simplest consolidation model and the one developed in this section, which has the greatest degree of sophistication, there are a total of five intermediate models. Each model has different basic variables. In order of increasing simplification, we will give the list of basic variables in each of the seven models. In what follows, it is evaluated for $\alpha = 1$, 2 if a superindex (α) is used. The seven models have the following corresponding independent variables: (1) $u_i^{(\alpha)}, p^{(\alpha)}, \bar{u}_i, \bar{p}$; (2) $u_i^{(\alpha)}, p^{(\alpha)}, \bar{u}_i$; (3) $u_i^{(\alpha)}, p^{(\alpha)}, \bar{p}$; (4) $u_i^{(\alpha)}, p^{(\alpha)}$; (5) $u_i^{(\alpha)}, p$; (6) u_i, $p^{(\alpha)}$; (7) u_i, p. As can be seen, there are a number of alternatives to model the same bi-laminated porous media. For a given practical problem, it seems difficult to assess *a priori* which of the possible models is the best. It may be more reliable to implement first the most complicated model and then to determine the most appropriate degree of sophistication. In the next section, a theory for jointed media will be studied. In the context of the family of models discussed above, it corresponds to model 6, which has five basic variables, u_i, $p^{(1)}$ and $p^{(2)}$.

4. MIXTURE EQUATIONS FOR JOINTED SATURATED POROUS MEDIA

Now we concentrate on a special case of the general bi-laminated saturated porous media: jointed saturated porous media (see Fig. 3). The material denoted by the superindex 2 will be referred henceforth to as the intact material, and the material labeled by 1 as the joint. Although the general framework established in the last section can directly be applied to the jointed medium, some features are worthy of special consideration.

To begin with, the hydraulic characteristics of the joint and the intact material are very different; the permeability of the joint may be higher than that of the intact material by

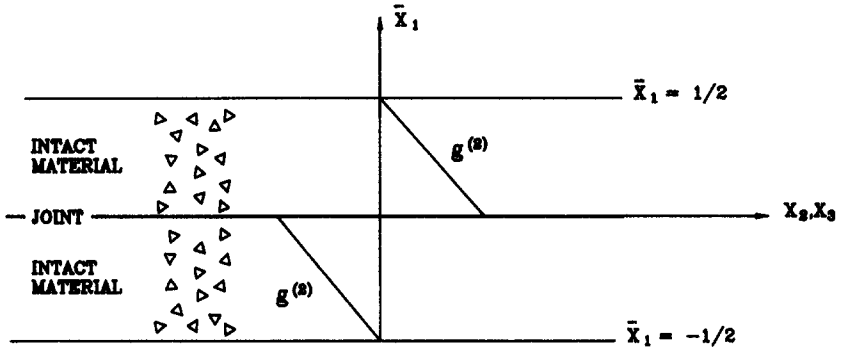

Fig. 3. Schematic view of jointed porous media at the microscopic level.

some orders of magnitude. Consequently a roughly averaged property does not reflect the true flow behavior, and distinction between the phase fluid pressures must be made in such a way that filling fluid follows two distinct flow paths. In order to facilitate the discussion, the permeability tensor $\kappa_{ij}^{(\alpha)}$ is assumed to have a transversely isotropic property

$$\kappa_{ij}^{(\alpha)} = \mathrm{diag}(\kappa_n, \kappa_t, \kappa_t)^{(\alpha)}, \tag{28}$$

where $\kappa_n^{(\alpha)}$ and $\kappa_t^{(\alpha)}$ are components in the directions normal and parallel to the layering. By adopting the assumption $\bar{p}_{,i} = 0$, we have from eqn $(26)_2$ that

$$\bar{q} = \frac{12k}{l^2}(p^{(1)} - p^{(2)}). \tag{29}$$

By using the above expression and eqn $(14)_3$, together with the assumptions $\dot{u}_{k,k} = \dot{\bar{p}} = 0$, we have from eqn (25) the expressions for $\bar{p}$:

$$\bar{p} = \frac{1}{d_3}(\kappa_n^{(2)} p_{,1}^{(2)} - \kappa_n^{(1)} p_{,1}^{(1)}), \tag{30}$$

where d_3 is defined in Appendix B. On the right-hand side of the above equation, $\kappa_n^{(2)} p_{,1}^{(2)}$ and $\kappa_n^{(1)} p_{,1}^{(1)}$ represent the average fluxes in the normal direction of lamina in pores and joint, respectively. So by the assumption that the micropressure $\bar{p}$ does not vary spatially, the normal flux difference between both phases remains uniform in space.

Generally, the stiffness of the joint is lower than that of the intact material, but this stiffness difference in the materials is not as relevant as the permeability difference. Moreover, as the volume fraction of the joint approaches zero, $\phi^{(1)} \approx 0$, and the joint is embedded in the intact medium, the macroscopically averaged deformation is mainly due to that of the intact material. Taking into account these special features, the mixture can be seen as a composite with average effective properties, without the necessity to make a distinction between the deformations in both constituents. Thus, if σ_{ij} and e_{ij} are the average stress and strain of the mixture, we have the average equilibrium equation

$$\sigma_{ji,j} = 0 \tag{31}$$

and the strain occurring in the intact material may be approximately equal to the average,

i.e. $e_{ij}^{(1)} \cong e_{ij}$. If the isotropic property is assumed for the intact material, eqn $(14)_1$ may be replaced by

$$\sigma_{ij}^{(2)} = \lambda e_{kk}\delta_{ij} + 2Ge_{ij} - \lambda\delta_{ij}\bar{u}_1 - G(\delta_{1j}\bar{u}_i + \delta_{i1}\bar{u}_j) - \zeta^{(2)}p^{(2)}\delta_{ij} \tag{32}$$

with λ and G being Lamé's elastic constants. On the other hand, the joint behavior is defined by a relation between the relative displacement $l\bar{u}_i$ and joint traction $\sigma_{1i}^{(1)}$ so eqn $(14)_1$ may be evaluated for the joint material as follows:

$$\sigma_{1i}^{(1)} = D_{ji}l\bar{u}_j - \zeta^{(1)}p^{(1)}\delta_{1i}, \tag{33}$$

D_{ij} is the joint stiffness (Goodman *et al.*, 1968). Although no difficult arises if nonlinear stiffness is introduced at this stage, it is preferable, for the sake of clarity, to assume an elastic joint property. To further simplify the presentation, a transversely isotropic joint behavior is assumed

$$D_{ij} = \text{diag}\,(D_n, D_t, D_t) \tag{34}$$

with D_n and D_t as normal and tangent joint stiffnesses. By making the assumptions $\bar{u}_{1,i} = \bar{u}_{i,1} = 0$ and $u_i^{(1)} = u_i^{(2)}$, together with the previous hypothesis that $\bar{p}_{,i} = 0$, eqn $(24)_1$ results in $\bar{\sigma}_{j,1} = 0$. Under these circumstances we have, starting from eqn (24), that $\sigma_{1i}^{(1)} = \sigma_{2i}^{(2)}$. This equilibrium condition, combined with eqns (32) and (33), leads to the following expressions for $\bar{u}_i$:

$$\begin{aligned} \bar{u}_1 &= (\lambda e_{kk} + 2Ge_{11} + \zeta^{(1)}p^{(1)} - \zeta^{(2)}p^{(2)})/d_1 \\ \bar{u}_2 &= 2Ge_{12}/d_2, \quad \bar{u}_3 = 2Ge_{13}/d_2, \end{aligned} \tag{35}$$

where d_1 and d_2 are given in Appendix B.

Moreover, the fluid mass change equation $(14)_2$ becomes

$$\begin{aligned} \phi^{(1)}m^{(1)} &= \zeta^{(1)}\bar{u}_1 + \eta^{(1)}p^{(1)} \\ \phi^{(2)}m^{(2)} &= \zeta^{(2)}e_{kk} - \zeta^{(2)}\bar{u}_1 + \eta^{(2)}p^{(2)}. \end{aligned} \tag{36}$$

These equations indicate that there are three sources contributing to the fluid mass change in the intact material: solid skeleton deformation, pore pressure and relative displacement of joint. In the joint, only the relative displacement and joint pressure induce the mass change. It is obvious from eqn (36) that a positive relative displacement—joint open—induces an increase of joint fluid mass but a decrease of pore fluid mass.

At this point all the necessary expressions for the microquantities are obtained. By inserting eqns (35), (29) and (30) in eqns (32), (33), (36) and $(14)_2$, we obtain the final constitutive equation whose compact form is given by

$$\begin{aligned} \{\sigma\} &= [C]\{e\} - \sum_{\alpha=1}^{2} \{\beta\}^{(\alpha)}p^{(\alpha)} \\ \phi^{(\alpha)}m^{(\alpha)} &= \{\beta\}^{(\alpha)\mathrm{T}}\{e\} + \alpha^{(\alpha)}p^{(\alpha)} - \alpha^{c}(\delta_{1\alpha}p^{(2)} + \delta_{2\alpha}p^{(1)}) \\ \phi^{(\alpha)}\{q\}^{(\alpha)} &= -[\kappa]^{(\alpha)}\{\partial p\}^{(\alpha)} - [\kappa^c]\,(\delta_{1\alpha}\{\partial p\}^{(2)} + \delta_{2\alpha}\{\partial p\}^{(1)}) \end{aligned} \tag{37}$$

in which it is defined that

$$\{\sigma\} = [\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{23}, \sigma_{31}]^T,$$

$$\{e\} = [e_{11}, e_{22}, e_{33}, 2e_{12}, 2e_{23}, 2e_{31}]^T,$$

$$\{q\} = [q_1, q_2, q_3]^T$$

and

$$\{\partial p\} = \{\partial p/\partial x_1, \partial p/\partial x_2, \partial p/\partial x_3\}^T.$$

The material constants $[C]$, $\{\beta\}^{(\alpha)}$, $\alpha^{(\alpha)}$, α^c, $[\kappa]$ and $[\kappa^c]$ are given in Appendix B.

In conclusion, the motion equations (31) and $(12)_2$, the constitutive equations (29) and (37), appropriate initial and boundary conditions define the consolidation problem in jointed saturated porous media. The basic variables are the displacement of solid skeleton, pore pressure and joint pressure. Once the solution of the mixture equations is obtained, the microstructural response may be recovered through eqns (21) and (23).

5. DISCUSSION

As stated in the Introduction, Aifantis and his co-workers (Wilson and Aifantis, 1982; Beskos and Aifantis, 1986; Khaled *et al.*, 1984) have developed a double porosity model to study consolidation behavior in a jointed fluid-saturated porous medium. As it was formulated within a different framework from the one used in the present work, it is interesting to compare the results. To begin with, the equations of motion derived in the two models are the same. The set of equations of motion consist of one equation of linear momentum balance (31) and two equations of mass conservation $(12)_2$. In the mass conservation equations, terms expressing mass exchange between the intact material and the joint appear, which are proportional to the fluid pressure difference.

The major difference between the two models is observed in the constitutive equations. The stress–strain relation $(37)_1$ is similar to those used in the previous model: the total stress is proportional to the strain, the pore pressure and the joint pressure. The matrix $[C]$ given in eqns (B1) and (B2) is the same as that proposed in the work of Murakami (1985). In a limiting case where the joint stiffness is very high, the stiffness for intact material is recovered. In contrast, when the stiffness of intact material is large compared with that of the joint, the medium can be seen as an ensemble of rigid blocks. On the other hand, whereas eqn $(37)_1$ is totally consistent with those previously obtained, in the other two constitutive equations, $(37)_2$ and $(37)_3$, additional cross or coupling terms appear that are not included in the previous model.

For the fluid mass change calculation, the cross coefficient α^c, included in the last term of eqn $(37)_2$, is not used in the previous work even in an improved version of the original model (Beskos, 1990) or other new application models (Cho *et al.*, 1991; Elsworth and Bai, 1992). The coupling effect indicates that the fluid mass change of the intact material is influenced not only by the pore pressure change but also by the joint pressure. In the same way, the fluid mass change in the joint is affected by both pore and joint pressures. The physical meaning behind this coupling effect is obvious. In the intact material, there are two sources making contributions to its fluid mass change, pore fluid pressure change and solid skeleton deformation; but the deformation is related, in turn, to the joint pressure, which is indicated by eqn $(35)_1$. Thus, any joint pressure change induces fluid mass change in the intact material. Conversely, any pore pressure change also causes fluid mass change in the joint material. In addition, it is interesting to observe that there is a negative sign before the cross term α^c, meaning that the fluid pressure change in one material (intact or joint) makes an opposite contribution to the fluid mass change in the other material (joint or intact). This can also easily be understood. Consider for example the intact material; with increasing pore pressure, the fluid mass content increases and the solid skeleton expands. At the same time, the joint tends to close and the fluid mass content in the joint decreases.

We can see that there are two key physical processes behind the macroscopic cross phenomena: the fluid–deformation coupling in porous materials, and the stress continuity

and displacement compatibility between the intact material and the joint. Both effects can easily be visualized from the expression for α^c in eqn (B5). α^c is related to the stiffnesses of both intact and joint materials through d_1 and to the deformation–pressure characteristics through $\zeta^{(1)}$ and $\zeta^{(2)}$. Quantitatively, α^c is of the same order as $\alpha^{(1)}$ and $\alpha^{(2)}$, so it is not insignificant. Returning to the model proposed by Aifantis and his co-workers, as it is based on the phenomenological mixture theory, the structure of constitutive equations has to be assumed *a priori*, so deciding what terms should be retained and what may be discarded is somewhat arbitrary. Although the cross terms can easily be included in the phenomenological mixture theory, they have been ignored. The appearance of the missing cross terms in the present work shows once again the advantage of a non-phenomenological approach in which equations result from rigorous microstructural considerations.

In the Darcy-type equation $(37)_3$, a cross term $[\kappa^c]$ that has not been included in the previous works emerges. This term, which comes from the flux continuity condition along the joint, indicates that the flux in one constituent is affected by pressure gradients of both phases. Originally, Darcy's law was established for a porous medium with single porosity. It is common practice to extend this law to treat multiphase flow [see, for example, Alder and Brenner (1988)], expressing the proportionality between the flux of one phase and the pressure gradient of the same phase. However, it has been pointed out (Alder and Brenner, 1988), without any theoretical or experimental justification, that cross terms may be expected. The results in the present work can be seen as some of the possible theoretical assessments on the subject. The cross terms emerge directly from the microstructural continuity conditions so a complete Darcy's law should include them.

The present model provides a more complete theory than those previously developed. In addition, the results offer an advantage for engineering use. The coefficients involved in the model can directly be calculated from *a priori* known characteristics of the media. This enhances the predictive capability of the model to a great extent. As pointed out above, the mass exchange term $\bar{q}$ expresses, at a microstructural level, the fluid flux across the joint in the direction normal to the layering. As shown in eqn (29), $\bar{q}$ is proportional to the fluid pressure difference, just as suggested empirically by Warren and Root (1963). The proportionality constant depends upon the average permeability in the normal direction of lamina k, which is in agreement with the definition of $\bar{q}$. Further, a value of 12 appears in the proportionality constant, which is consistent with other plate theories. As can also be seen, $\bar{q}$ depends on the square of joint spacing l; the smaller l, the more mass exchange. In the limiting case for which l approaches zero, in order that a finite mass exchange occurs, the fluid pressures in both phases should be equal recovering a consolidation model with single porosity. The mass exchange $\bar{q}$ is not the only term including the joint spacing l. All the mixture properties related to the deformation effects depend upon the spacing l, but now to a lesser extent because the linear rather than quadratic term of l is included. l-dependence of the mixture is also relevant when one wants dynamic dispersion phenomena to be accurately modeled (Murakami *et al.*, 1981; Murakami, 1985; Murakami and Hegemier, 1989; Murakami and Toledano, 1990).

The flow problem in a jointed saturated porous medium is of two-phase type. The present theory is not sufficiently general to deal with any two-phase flow problem. The two-flow problem considered here is the simplest one because the flow is miscible. One of the characteristics of two miscible fluid-saturated porous media is that the deformation–pressure coefficients are symmetric. This is certainly true, as can be seen from eqns $(37)_1$ and $(37)_2$ in which the same vector $\{\beta\}^{(\alpha)}$ appears in both equations. For a general immiscible two-flow problem, where capillarity plays an important role, the symmetric structure concerning the coefficients $\{\beta\}$ may be lost and a more general theory is required. More details on this subject are found in Li and Li (1992) and Li (1994).

To close the discussion of constitutive equations, let us consider a limiting case in which the fluid pressures take the same value, thus recovering a model for porous media with single porosity. In this case we have from eqn (37) that

$$\{\sigma\} = [C]\{e\} - \{\beta\}p, m = \{\beta\}^{\mathrm{T}}\{e\} + \alpha p, \{q\} = -[\kappa]\{\partial p\} \tag{38}$$

with $m = \Sigma \phi^{(\alpha)} m^{(\alpha)}$, $\{\beta\} = \Sigma \{\beta\}^{(\alpha)}$, $\alpha = \Sigma \alpha^{(\alpha)} - 2\alpha^c$, $\{q\} = \Sigma \phi^{(\alpha)} \{q\}^{(\alpha)}$, and $[\kappa] = \Sigma [\kappa]^{(\alpha)} + 2[\kappa^c]$. A series of effective properties for the mixture are obtained:

$$\{\beta\} = [\zeta^{(2)} + (\zeta^{(1)} - \zeta^{(2)})(\lambda + 2G)/d_1, \zeta^{(2)} + (\zeta^{(1)} - \zeta^{(2)})\lambda/d_1, \zeta^{(2)} + (\zeta^{(1)} - \zeta^{(2)})\lambda/d_1, 0, 0, 0]^T$$

$$\alpha = \eta^{(2)} + (\zeta^{(1)} - \zeta^{(2)})^2/d_1$$

$$[\kappa] = \text{diag}\left(k, \sum_{\alpha=1}^{2} (\phi\kappa_s)^{(\alpha)}, \sum_{\alpha=1}^{2} (\phi\kappa_s)^{(\alpha)}\right), \tag{39}$$

where k is given in eqn (26). Note that if the coefficients $\zeta^{(1)}$ and $\zeta^{(2)}$ are not mismatched, the effective coefficients $\{\beta\}$ and α are defined only by the intact properties. The form of permeability tensor $[\kappa]$ defined in eqn (39) is well known.

The methodology employed in the present work to obtain mixture equations is essentially the same as developed in Murakami *et al.* (1981), Murakami (1985), Murakami and Hegemier (1989) and Murakami and Toledano (1990). Such a mixture approach leads to a higher-order theory. However, phenomenological mixture models generally do not include terms of higher order. In order to obtain consistent results in both models, it is necessary to eliminate the variables of higher order. This paper presents a hierarchy of models with different degrees of sophistication first and then establishes a procedure to develop simplified theories by making pertinent assumptions about the quantities of higher order. This makes the obtained model transparent because we are aware of hypotheses made in each stage of development. For instance, for the widely used model of consolidation with double porosity, we have to accept the assumptions that the flux difference in both constituents in the direction normal to the layering is constant [eqn (30)] and the microdiffusion process is discarded ($\dot{\bar{p}} = 0$).

6. CONCLUSIONS

In this study, motion and constitutive equations for a bi-laminated saturated porous medium are formulated within the framework of non-phenomenological mixture theory. Special attention is devoted to a fluid-saturated porous medium with a single joint system. The method used to derive mixture equations is based on an asymptotic analysis of a periodic structure with multiple scales. The Biot consolidation problem is originally coupled, transient. The asymptotic analysis proposed here allows us to replace it by various uncoupled, steady-state problems which are defined at the microscopic level. This is quite encouraging because independent solutions for elasticity and conduction, that are well known, can be combined in a straightforward way to furnish useful results.

A hierarchy of bi-laminated models with different degrees of sophistication is presented. The model with the greatest degree of sophistication embraces eqns (12), (14), (24), (25) and (26). By appropriate assumptions about the micro quantities, the model can be simplified. For the jointed media, the constitutive equations are derived in such a way that they are consistent with those used in a phenomenological theory. The main results for the jointed media are summarized in eqns (31), $(12)_2$, (29) and (37). Compared with previous models, the motion equations are essentially the same but the constitutive equations are very different except those describing solid skeleton deformation. The most distinctive feature of the present model is that it provides some cross terms in the constitutive equations for fluid mass change and flux. The fluid mass change (or flux) depends upon the fluid pressure (or pressure gradient) in intact material as well as in joint material. Due to their magnitudes, in most cases these cross terms cannot be ignored. Although these conclusions are drawn from a special porous medium, their validity for other more complicated cases such as random media is expected, because the physical processes behind the cross effects always take place.

The novel results obtained here may have far-reaching consequences for future theoretical modeling and experimental programs in two-phase fluid-filled porous media. For instance, the generally accepted Darcy's law for the porous medium with single porosity

cannot directly be extended to that with double porosity. Special experimental difficulty may arise when the cross conductivity coefficients must be determined directly. Also, the cross terms related to the fluid mass change cannot be measured easily. Usual techniques utilizing concepts of different time and geometrical scales may have to be modified. More theoretical studies may provide some guidance on these questions.

The geometry of the problem treated in this paper is relatively simple. These results will be extended to more complicated jointed media such as with continuous or discontinuous staggered joints or random media. Also, the methodology developed in the present work will be useful to study the diffusion-induced stress problem in composite materials with grain boundaries.

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APPENDIX A: REISSNER'S VARIATIONAL PRINCIPLE FOR A BINARY MEDIUM

The modified Reissner's variational principle (Reissner, 1986) takes the following form of virtual work

$$\int_\Omega \Bigg[\sum_\alpha \int_{\Gamma^{(\alpha)}} (\delta \hat{u}^{(\alpha)}_{i,j} \hat{\sigma}^{(\alpha)}_{ij} + \delta \hat{p}^{(\alpha)}_{,i} \hat{q}^{(\alpha)}_i + \delta \hat{\sigma}^{(\alpha)}_{ij} (\hat{e}^{(\alpha)}_{ij} - (C^{-1})^{(\alpha)}_{ijkl} (\hat{\sigma}^{(\alpha)}_{kl} + \zeta^{(\alpha)} \hat{p}^{(\alpha)} \delta_{kl}))$$

$$+ \delta \hat{q}^{(\alpha)}_i (\hat{p}^{(\alpha)}_{,i} + (\kappa^{-1})^{(\alpha)}_{ij} \hat{q}^{(\alpha)}_j)) \mathrm{d}v + \frac{1}{l} \sum_\alpha (-1)^\alpha \int_A (\delta \hat{u}^{(\alpha)}_i \hat{\sigma}^{(\alpha)}_{1i} + \delta \hat{\sigma}^{(\alpha)}_{1i} \hat{u}^{(\alpha)}_i$$

$$+\delta\hat{p}^{(\alpha)}\hat{q}_1^{(\alpha)}+\delta\hat{q}_1^{(\alpha)}\hat{p}^{(\alpha)})v_1^{(1)}\,\mathrm{d}s\Big]\mathrm{d}\Omega=\int_\Omega\Big[\sum_\alpha\int_{v^{(\alpha)}}\delta\hat{p}^{(\alpha)}\hat{m}^{(\alpha)}\,\mathrm{d}v\Big]\mathrm{d}\Omega$$

$$+\int_{\Gamma_\sigma}\Big[\sum_\alpha\int_{v^{(\alpha)}}\delta\hat{u}_i^{(\alpha)}S_i^{(\alpha)}\,\mathrm{d}v\Big]\mathrm{d}\Gamma,$$

$$+\int_{\Gamma_q}\Big[\sum_\alpha\int_{v^{(\alpha)}}\delta\hat{p}^{(\alpha)}Q^{(\alpha)}\mathrm{d}v\Big]\mathrm{d}\Gamma, \tag{A1}$$

where $v_1^{(1)}$ is the outward normal of the phase 1 on the interfacial area. The trial deformation tensor $\hat{e}_{ij}^{(\alpha)}$ is defined as

$$2\hat{e}_{ij}^{(\alpha)}=\hat{u}_{i,j}^{(\alpha)}+\hat{u}_{j,i}^{(\alpha)}.$$

All the other quantities in eqn (A1) have been defined in Sections 2 and 3. After applying the Gauss's theorem to eqn (A1), we finally have

$$\int_\Omega\Big[\sum_\alpha\int_{v^{(\alpha)}}(\delta\hat{u}_i^{(\alpha)}\hat{\sigma}_{ij,j}^{(\alpha)}+\delta\hat{p}^{(\alpha)}(\hat{q}_{i,i}^{(\alpha)}+\hat{m}^{(\alpha)})$$

$$-\delta\hat{\sigma}_{ij}^{(\alpha)}(\hat{e}_{ij}^{(\alpha)}-(C^{-1})_{ijkl}^{(\alpha)}(\hat{\sigma}_{kl}^{(\alpha)}+\zeta^{(\alpha)}\hat{p}^{(\alpha)}\delta_{kl}))-\delta\hat{q}_i^{(\alpha)}(\hat{p}_{,i}^{(\alpha)}+(\kappa^{-1})_{ij}^{(\alpha)}\hat{q}_j^{(\alpha)}))\,\mathrm{d}v$$

$$+\frac{1}{l}\sum_\alpha(-1)^{\alpha+1}\int_A(\delta\hat{u}_i^{(\alpha)}\hat{\sigma}_{1i}^{(\alpha)}+\delta\hat{\sigma}_{1i}^{(\alpha)}\hat{u}_i^{(\alpha)}+\delta\hat{p}^{(\alpha)}\hat{q}_1^{(\alpha)}+\delta\hat{q}_1^{(\alpha)}\hat{p}^{(\alpha)})v_1^{(1)}\,\mathrm{d}s\Big]\mathrm{d}\Omega$$

$$=\int_{\Gamma_\sigma}\Big[\sum_\alpha\int_{v^{(\alpha)}}\delta\hat{u}_i^{(\alpha)}(\hat{\sigma}_{ij}^{(\alpha)}n_j-S_i^{(\alpha)})\mathrm{d}v\Big]\mathrm{d}\Gamma+\int_{\Gamma_q}\Big[\sum_\alpha\int_{v^{(\alpha)}}\delta\hat{p}^{(\alpha)}(\hat{q}_i^{(\alpha)}n_i-Q^{(\alpha)})\,\mathrm{d}v\Big]\mathrm{d}\Gamma. \tag{A2}$$

Equation (A2) is used to derive all the mixture equations as well as boundary conditions.

APPENDIX B: CONSTITUTIVE EQUATIONS FOR THE JOINTED POROUS MEDIA

The mixture stiffness parameters are given by three tensors $[C]$ and $[C^c]$:

$$[C]=\begin{bmatrix}a&b&b&0&0&0\\b&c&d&0&0&0\\b&d&c&0&0&0\\0&0&0&f&0&0\\0&0&0&0&G&0\\0&0&0&0&0&f\end{bmatrix} \tag{B1}$$

with

$$a=(\lambda+2G)(1-(\lambda+2G)/d_1),\quad b=\lambda(1-(\lambda+2G)/d_1)$$
$$c=(\lambda+2G)(1-\lambda/d_1),\quad d=\lambda(1-\lambda/d_1)$$
$$f=G(1-G/d_2), \tag{B2}$$

where

$$d_1=lD_n+\lambda+2G,\quad d_2=lD_t+G. \tag{B3}$$

The constants relating the pore pressure and total stress are given by two vectors $\{\beta\}^{(1)}$, $\{\beta\}^{(2)}$:

$$\{\beta\}^{(1)}=\zeta^{(1)}[(\lambda+2G)/d_1,\lambda/d_1,\lambda/d_1,0,0,0]^{\mathrm{T}}$$
$$\{\beta\}^{(2)}=\zeta^{(2)}[1-(\lambda+2G)/d_1,1-\lambda/d_1,1-\lambda/d_1,0,0,0]^{\mathrm{T}}. \tag{B4}$$

The constants relating the fluid mass change and pore pressure are given by three scalars $\alpha^{(1)}$, $\alpha^{(2)}$ and α^c:

$$\alpha^{(1)}=\eta^{(1)}+\zeta^{(1)}\zeta^{(1)}/d_1,\quad \alpha^{(2)}=\eta^{(2)}+\zeta^{(2)}\zeta^{(2)}/d_1,\quad \alpha^c=\zeta^{(1)}\zeta^{(2)}/d_1. \tag{B5}$$

Finally, the hydraulic conductivity constants are given by three tensors, assuming a transversely isotropic case:

$$[\kappa]^{(\alpha)} = \operatorname{diag}(\kappa_n^{(\alpha)}(\phi^{(\alpha)} - \kappa_n^{(\alpha)}/d_3), \phi^{(\alpha)}\kappa_t^{(\alpha)}, \phi^{(\alpha)}\kappa_t^{(\alpha)})$$

$$[\kappa^c] = \operatorname{diag}(\kappa_n^{(1)}\kappa_n^{(2)}/d_3, 0, 0), \tag{B6}$$

where

$$d_3 = \sum_{\alpha=1}^{2} \kappa_n^{(\alpha)}/\phi^{(\alpha)}. \tag{B7}$$

Equation (B6) may be further reordered such that

$$[\kappa]^{(\alpha)} = \phi^{(\alpha)} \operatorname{diag}(\phi^{(\alpha)}k, \kappa_t^{(\alpha)}, \kappa_t^{(\alpha)})$$

$$[\kappa^c] = \operatorname{diag}(\phi^{(1)}\phi^{(2)}k, 0, 0), \tag{B8}$$

where k is given in eqn (26).